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Waves in parallel or swirling stratified shear flows†

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The dispersion relations for infinitesimal internal gravity waves (A) and axisymmetric waves in swirling streams (B) are considered. In both cases the mainstream may be sheared and density stratified in the transverse (vertical in case A, radial in case B) direction. The following results are proved for either case: If the maximum speed W_{max} (or minimum speed W_{min}) (in a meridian plane in case B) of the mainstream occurs at an interior point in the fluid, then the phase speed of any mode takes all values from the W_{max} (or W_{min}) to $+\infty$ ($-\infty$) as the overall Richardson number λ^2 varies from 0 to ∞. If W_{max} (W_{min}) is attained at a boundary point with finite rate of strain, there is a positive non-zero critical Richardson number below which one or both branches of the dispersion relation terminate. These results employ variational methods and correct erroneous results concerning problem B stated in Chandrasekhar's treatise on hydrodynamic stability. Furthermore, bounds are given on the group velocity for both branches of the dispersion relation. From these bounds it is shown that in the absence of reversals of the mainstream ($W_{min} > 0$) upstream propagation of wave energy is impossible whenever upstream propagation of constant phase surfaces is impossible.

1. Introduction

Qualitative features of the dispersion relation for waves in inviscid density-statified streaming fluids are considered. The mathematical (and physical) similarities of wave propagation in density-stratified fluids under gravity (case A, internal gravity waves) on sheared currents and on streaming vortices (case B, where centrifugal force replaces gravity as the restoring force) are relatively well known. In the present paper we encompass both systems and also allow density stratification (in the radial direction) in the case of swirling flows. Sketches of the problems to be considered are shown in figure 1.

The waves we are interested in are those corresponding to the real discrete spectrum of the normal modes of the system; it is this part of the spectrum that contributes neutrally stable propagating wave modes. The phase speeds for these waves are either larger than the maximum flow speed, or smaller than the minimum flow speed. When the local Richardson number (defined in § 2) exceeds $\frac{1}{4}$ everywhere, than these flows are stable to infinitesimal disturbances‡ (Miles 1961; Howard & Gupta 1962; Leibovich

† This paper is dedicated to the memory of my friend Stephen A. Thau, whose unexpected death was a sad and irreplaceable loss to the mechanics community.

‡ Maslowe (1977) shows the possibility of subcritical instability to sufficiently large disturbances when the local Richardson number exceeds $\frac{1}{4}$.

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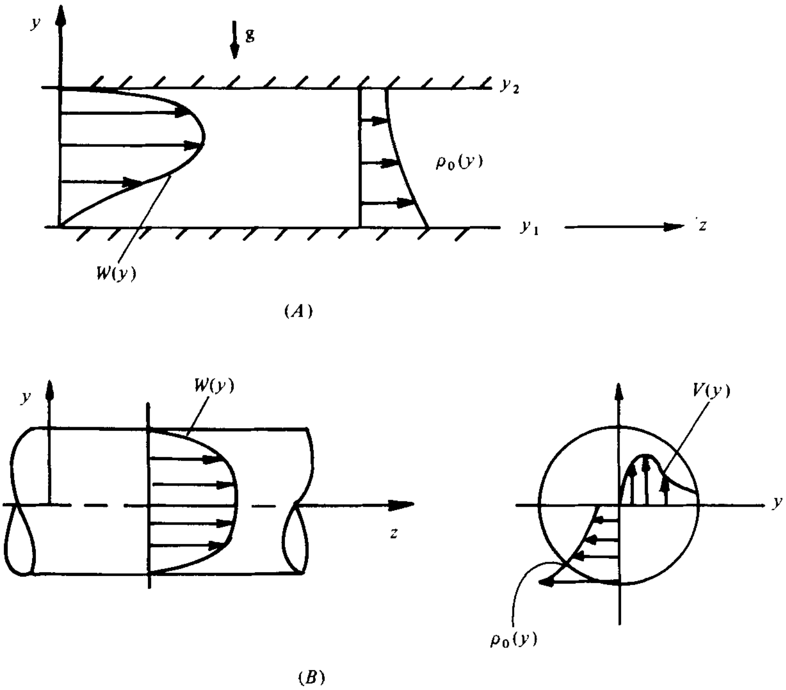

FIGURE 1. Illustrations of the two classes of problems considered. (*A*) Internal gravity waves in a fluid confined between parallel planes at $y = y_1$ and $y = y_2$. (*B*) Swirling flow in a tube; the fluid may be density straified, and confined in an annulus between radii $y_1 \geqslant 0$ and $y_2 > y_1$.

1969), and the only infinitesimal disturbance components are these waves. If the local Richardson number is less than $\frac{1}{4}$, stability cannot be guaranteed and the flow may become unstable. Whether or not instability actually occurs, propagating neutral modes may still be possible, as shown by Banks, Drazin & Zaturska (1976) for special cases involving internal gravity waves on sheared currents.

A disturbance streamfunction can be defined for propagating neutral modes, and assumes the form

$$\Psi = \psi(y) \exp\{i[k(z-ct)]\},$$

where c is the phase speed, and k the wavenumber. If the local Richardson number is $\Phi(y)$, then a device often used is to introduce an 'overall Richardson number' (denoted by λ^2 herein) and to write $\Phi(y) = \lambda^2 \phi(y)$. Thus $\phi(y)$ is a normalized function describing the shape of $\Phi(y)$, and the parameter λ^2 sets the scale of Φ. For a given shape $\phi(y)$, the dispersion relation may then be written as $\mathscr{F}(c, k, \lambda^2) = 0$. For fixed values of k and λ^2, more than one value of c may satisfy the dispersion relation. If admissible values are labelled c_n, $n = 1, 2, \ldots$ each eigenvalue c_n corresponds to an eigenfunction $\psi_n(y)$ which describes the vertical structure of that wave mode. The purpose of this paper is to study the characteristics of these waves for both physical systems *A* and *B* as a function of overall Richardson number for arbitrary current speeds [denoted by $W(y)$], arbitrary statically stable density (denoted by $\rho_0(y)$ and $\rho_0' < 0$) and, in case *B*, arbitrary azimuthal speeds [denoted by $V(y)$].

Banks *et al.* (1976) have covered much of the ground of this paper for problem *A* (neglecting the inertial effects of density stratification, a step that is unnecessary with present treatment). By constructing detailed numerical and analytical solutions for special cases (ρ_0'/ρ_0 = constant, and various mainstream profiles $W(y)$), they found the following.

(i) If the maximum W_{max} (or minimum W_{min}) of $W(y)$ occurs in the interior of the fluid, then an infinite number of propagating neutral modes exist for all positive values of λ^2, and the phase speed $c \downarrow W_{max}$ ($c \uparrow W_{min}$) as $\lambda^2 \to 0$.

(ii) $\lambda^2 \to \lambda_0^2 \neq 0$ as c approaches an exteme value of the flow speed (i.e. either W_{max} or W_{min}) that is attained at a boundary of the flow.

Chandrasekhar (1960), using two variational principles, claimed to have demonstrated (i) for case B (with ρ_0 = constant) without regard to the position of the extremum in W. In fact, Chandrasekhar argues from his analysis that the shear $W'(y)$ does not play a role in determining stability of these flows, and that stability is assured if Rayleigh's criterion that the square of the circulation (y^2V^2, if y is the radius) increases outwards is satisfied. Howard & Gupta (1962), without identifying the flaw in the argument, point out that this conclusion must be incorrect. Here we use variational principles also and, correcting and extending Chandrasekhar's work, rigorously prove (i) and (ii) for general density and velocity profiles for both cases A and B.

Bell (1974)† has studied the behaviour of the dispersion relation for internal gravity waves with shear, for given $\Phi(y)$, as a function of n (k fixed) and as a function of k (n fixed). Using Sturmian methods, he shows that (a) if $\Phi(y) > \frac{1}{4}$ for all y of interest, a denumerably infinite set of normal modes exist for each fixed k corresponding to eigenvalues c which have a finite maximum (minimum) and converge to W_{max} (W_{min}) as n increases indefinitely, and (b) for fixed n, $|c|$ is a decreasing function of k, which tends to $k^{-1}\Phi_{max} + W_{max}$ ($k^{-1}\Phi_{max} + |W_{min}|$) as k increases indefinitely. The results of the present paper therefore complement Bell's work.

We also consider the group velocity of the propagating neutral modes. Benjamin (1962) demonstrated (for case B with ρ_0 = constant) that the group velocity of standing waves (where the smallest possible phase speed $c = 0$) is positive (i.e. in the downstream direction). Thus no disturbance source in a swirling stream can produce a stationary wave pattern upstream of the source, and Benjamin thereby ruled out Squire's (1960) explanation of vortex breakdown (but see Leibovich 1978 for discussion of this point). Benjamin's own theory of vortex breakdown centres around a classification of vortex flows as subcritical (where the minimum phase speed is negative, i.e. constant phase surfaces can propagate upstream) or supercritical (where only downstream propagation of constant phase surfaces is possible). While this classification based upon phase speeds is adequate for Benjamin's theory (which contemplates phenomena that are stationary in time) it does leave open the question of whether (non-stationary) upstream propagation of wave packets is possible under the conditions defined as supercritical. In §4 we establish bounds on the group velocity that, among other things, answers this question in the negative, provided the axial velocity $W(y)$ exhibits no reversals of direction. In particular, we show for cases A and B that, for any given mode, the group velocity for the lower branch ($c < W_{min}$) of dispersion relation is in half-open interval $[c, W_{max})$, while the group velocity for the upper branch ($c > W_{max}$) is in the half-open interval $(W_{min}, c]$.

† I am indebted to a referee for bringing Bell's work to my attention.

2. Governing equations

We consider the wave propagation characteristics of incompressible, inviscid, density-stratified flows. The fluid may be in motion parallel to an axis; or may be in motion having symmetry, and an azimuthal velocity component, about the axis.

In both cases, the distinguished axis is taken to be the z-axis. In the parallel flow case (which will be denoted 'A'), we adopt a Cartesian (x, y, z) co-ordinate system and assume the undisturbed velocity vector is of the form

$$\mathbf{q} = (0, 0, W(y)),$$

where the y co-ordinate decreases in the direction of the gravitational acceleration. In the axisymmetric case (case 'B'), we use a cylindrical (y, θ, z) co-ordinate system, in which the radial distance from the symmetry axis is y, and assume the undisturbed fluid velocity vector is

$$\mathbf{q} = (0, V(y), W(y))$$

and gravity is ignored. In both cases, the undisturbed fluid density is

$$\rho = \rho_0(y)$$

and we assume that the undisturbed flow (i.e., ρ_0, W, and V) depends smoothly upon y.

If attention is restricted in case A to two-dimensional perturbations,† independent of x, and in case B to axisymmetric perturbations, then a streamfunction ψ is available. For both cases, normal modes can be investigated by writing the perturbation streamfunction in the form

$$\Psi = y^{\nu}[W(y) - c]\,\chi(y) \exp{[ik(z - ct)]},$$

where $\nu = 0$ for case A and $\nu = 1$ for case B, and the function $\chi(y)$ satisfies the equation (cf. Miles (1961) for case A, and Leibovich (1969) for case B)

$$D[\rho_0(y)\,(W-c)^2 D_* \chi] - \rho_0 k^2 (W-c)^2 \chi + \rho_0 \Phi \chi = 0, \qquad (1)$$

where

$$D = d/dy, \quad D_* = D + \nu/y$$

and

$$\Phi = \begin{cases} -g\rho_0^{-1} D\rho_0 & \text{for case } A, \\ y^{-3}\rho_0^{-1} D(\rho_0 y^2 V^2) & \text{for case } B, \end{cases}$$

g is the gravitational acceleration, and Φ will be assumed positive everywhere in the flow. (The definition of Φ for case B in Leibovich (1969) is ρ_0 times the definition used here.) We will assume the fluid to be confined by walls at $y = y_1$ and $y = y_2 > y_1$, so

$$\chi(y_1) = \chi(y_2) = 0. \qquad (2)$$

In case B, the walls at fixed y confine the fluid to a cylindrical annulus, but the inner wall may be absent (i.e. $y_1 = 0$).

The problem posed in (1) and (2) for case B has been considered by Chandrasekhar (1961) and by Howard & Gupta (1962) in the important special case of constant density. The extension of Howard & Gupta's main results to the present case B is straightforward, and has been carried out by Leibovich (1969). In particular, it can

† This entails no loss of generality. The normal mode problem for a wave propagating in the x, z plane with wavenumber vector $\mathbf{k} = k\,(\sin\theta, \cos\theta)$ is reduced, by a co-ordinate rotation, to a two-dimensional problem in the plane determined by the y axis and $\mathbf{k}$; the reduced problem is identical to (1) with $W(y)$ replaced by $W(y)\cos\theta$.

be shown that stability to infinitesimal disturbances is assured if the Richardson number

$$J(y) \equiv \Phi/(DW)^2 \geqslant \tfrac{1}{4} \quad (3)$$

everywhere in (y_1, y_2), and the identical result is shown by Miles (1961) for case A. We will consider all possible flows with $J > 0$, including those that may prove unstable, but we restrict attention to that part of the spectrum (e.g. set of c for k fixed) that corresponds to propagating neutral modes (defined in the next section).

Let the maximum value of W in $[y_1, y_2]$ be denoted W_{max}, and the minimum value W_{min}, so

$$W_{min} \leqslant W(y) \leqslant W_{max}$$

in $[y_1, y_2]$.

3. Qualitative features of the spectrum of propagating neutral modes

Propagating neutral modes (neutrally stable modes not adjacent to unstable modes), when they occur, have c real and either $c > W_{max}$, or $c < W_{min}$, so that the governing differential equation (1) is non-singular. When the flow is stable no other modes are possible; this is assured when the stability criterion (3) is satisfied, but may be true for smaller Richardson numbers also. Even when the flow is unstable, propagating neutral modes may still exist, as described by Banks *et al.* (1976) for the plane parallel stratified flow case.

The spectrum of problem (1) is usually discussed as a function, first of all, of wavenumber k for fixed Φ. We refer to this as the primary eigenvalue problem. On the other hand, the behaviour for k fixed and c prescribed, and variable Φ has been an alternative approach. Here one assumes

$$\Phi(y) = \lambda^2 \phi(y),$$

where $\phi(y)$ prescribes the functional form of Φ. Here λ^2 is a positive normalizing parameter (an 'overall Richardson number') that is sought as the eigenvalue when c and k are fixed. We refer to this as the secondary eigenvalue problem.

Chandrasekhar (1961, pp. 368–9) describes a variational principle for the phase speed c for case B (with ρ_0 = constant, but the arguments are not changed in case B if ρ_0 is variable, nor would they change for case A). This principle is associated with the primary eigenvalue problem. As Chandrasekhar points out, for real phase speeds c, the variational principle is valid only for propagating neutral modes. For the propagating neutral modes to which it applies, the variational principle appears to show that there are two and only two real values of c for fixed λ^2; one is greater than W_{max} and the other smaller than W_{min}. The validity of this variational principle, and the conclusions that have been drawn from it, will be discussed later.

From the secondary eigenvalue problem, which is in standard Sturm–Liouville form, one can deduce that an infinite number of real, positive λ^2 exist for each value of $c > W_{max}$, and for each value of $c < W_{min}$ by employing Sturmian theory. Each of these eigenvalues, which may be ordered, i.e. $\lambda_1^2 < \lambda_2^2 < \ldots$, corresponds to a distinct *mode* of oscillation: the eigenfunctions $\chi_1, \chi_2, \ldots$ corresponding to each mode can be identified by the number of zeros attained in the interior of the interval, with the nth eigenfunction displaying $n-1$ internal zeros. Thus one can trace the continuous variation of the nth eigenvalue $\lambda^2 = \lambda_n^2(c)$ as c varies continuously, since it always corresponds to that eigenfunction with $n-1$ internal zeros.

If k is held fixed and equation (1) is differentiated with respect to λ^2, we find that the function

$$f(y;\lambda^2) \equiv \partial\chi(y;\lambda^2)/\partial\lambda^2$$

satisfies an inhomogeneous differential equation with the same operator as (1). Since $\chi(y;\lambda^2)$ satisfies the homogeneous problem, a solution for f is possible only if the orthogonality condition

$$2\lambda^2(dc/d\lambda^2) = -\int_{y_1}^{y_2} \rho_0(W-c)^2 Qy^\nu\, dy \bigg/ \int_{y_1}^{y_2} \rho_0(W-c)\, Qy^\nu\, dy \tag{4}$$

where
$$Q \equiv (D_*\chi)^2 + k^2\chi^2 \tag{5}$$

is satisfied (here we have used the differential equation to make the replacement $\lambda^2 \int \rho_0 \phi\chi^2 y^\nu dy = \int (W-c)^2 \rho_0 Qy^\nu dy$). From (4) we conclude (Chandrasekhar gives a different and more involved proof) that c decreases with increasing λ^2 for all $c < W_{\min}$ (we call this the lower branch; c is regarded here as a function of λ^2 for k fixed) and c increases with increasing λ^2 for $c > W_{\max}$ (the upper branch). This shows, by the inverse function theorem, that the function $\lambda_n^2(c)$ determined from Sturm–Liouville theory to be the nth eigenvalue, can be inverted to give two distinct branches for the function $c_n(\lambda^2)$, with the upper branch (the superscript (u)) having $c_n^{(u)} > W_{\max}$ and the lower branch (denoted by the superscript (l)) having $c_n^{(l)} < W_{\min}$. Note that $c_n^{(u)}$ or $c_n^{(l)}$ corresponds to the nth mode of propagation.

It is well known that the functional

$$\lambda^2(c) = \int_{y_1}^{y_2} \rho_0(W-c)^2 Qy^\nu\, dy \bigg/ \int_{y_1}^{y_2} \rho_0 \phi\chi^2 y^\nu\, dy \tag{6}$$

achieves a minimum for solutions of the secondary eigenvalue problem. It is easy to see from this variational principle that for any given mode $\lambda^2 = O(|c|^2)$ asymptotically as $|c|^2 \to \infty$.

Chandrasekhar asserts that the variational principal (6) implies that $\lambda^2 \to 0$ as $c \downarrow W_{\max}$, or as $c \uparrow W_{\min}$, which would imply that all physically possible values of λ^2 correspond to values of c outside the range of W. The assertion that $\lambda^2 \to 0$ as $c \to W_{\max}$ or $c \to W_{\min}$ is neither obvious from the inspection of the variational principle, nor, as detailed calculations by Banks *et al.* (1976) for special cases suggest, is it always true. In that class of flows for which the claim is true, one would be tempted to follow Chandrasekhar's reasoning and conclude that such flows are stable provided Φ is positive. For each positive Φ, two real c exist outside the range of W and, by the first variational principle, only two values of c outside the range of W are possible. It thus appears that all possibilities are exhausted, and all possible solutions are propagating neutral modes.

This conclusion, however, is known to be incorrect. One fallacy (another is suggested at the end of this section) in the reasoning seems to be this: in the variational principle for λ^2 one must prescribe values of c, and the variational principle is meaningless when the c prescribed is complex. Thus, although a given value of λ^2 may be associated with two real c describing propagating neutral modes, complex values of c are not excluded for sufficiently small λ^2.

(*a*) *If an extreme value of* $W = W_{\text{extreme}}$ *occurs at an interior point, then* $\lambda^2(c) \to 0$ *as* $c \to W_{\text{extreme}}$. We now return to the claim that $\lambda^2 \to 0$ as $c \downarrow W_{\max}$ or as $c \uparrow W_{\min}$.

Chandrasekhar does not provide a proof of this: in fact, it is only obvious for $W(y) = \text{constant}$. If the overall maximum speed W_{max} (or minimum speed W_{min}) occurs at an interior point of the flow, it can be proved quite generally† that $\lambda^2 \to 0$ as $c \downarrow W_{max}$ (or $c \uparrow W_{min}$). Consider the function

$$g_\epsilon(y) \equiv \frac{(y-y_1)(y_2-y)}{((y-y_0)^2+\epsilon^2)^{\frac{1}{2}}}, \tag{7}$$

where y_0 is a point in the interior of (y_0, y_1) where W achieves either its maximum or minimum. To be definite, we assume the extreme value is a minimum: an essentially identical analysis holds if a maximum is considered. Then, if y_0 is an interior point,

$$W = W_{min} + K^2(y-y_0)^2 + O(y-y_0)^3$$

near $y = y_0$, where $K^2 = \partial^2 W/\partial y^2|_{y=y_0}$. If we define the parameter ϵ in (7) to be

$$\epsilon \equiv W_{min} - c$$

then $W - c = \epsilon + K^2(y-y_0)^2 + O(y-y_0)^3$ near $y = y_0$. Consider the functional (6) for λ^2. As $\epsilon \to 0$, the numerator of (6) is $O(\epsilon^0)$, but the denominator is

$$\pi\epsilon^{-1} y_0^\nu \phi(y_0)(y_0-y_1)^2(y_2-y_0)^2 + O(\epsilon^0),$$

so that

$$\lambda^2 = O(\epsilon) \tag{8}$$

as $\epsilon \to 0$. Since λ^2 is the minimum over all admissible (twice differentiable functions $\chi(y)$ satisfying the boundary conditions) the present demonstration shows that $c \to W_{min}$ implies $\lambda^2 \to 0$. In fact, the present estimate suggests that $c = W_{min} + O(\lambda^2)$ as $\lambda^2 \to 0$, in agreement with Banks *et al.* (1976, where the notation is $\lambda^2 \equiv J$).

We have thus shown that the lowest eigenvalue $\lambda_1^2(c) \to 0$ as $\epsilon \to 0$ for all flows in which the extreme values of $W(y)$ are confined to the interior of the flow domain. Recall that an infinite number of modes exist for regular Sturm–Liouville problems such as this, with eigenvalues that can be ordered $\lambda_1^2(c) < \lambda_2^2(c) < \ldots \lambda_n^2(c) < \ldots$. We can now show that, for any fixed mode n and wavenumber k, $\lambda_n^2(c) \to 0$ as $c \uparrow W_{min}$ ($c \downarrow W_{max}$) if W_{min} (W_{max}) occurs at an interior point. The analysis is carried out only for $\lambda_2^2(c)$, but the generalization to higher modes is clear.

Suppose the lowest eigenfunction corresponding to $\lambda_1^2(c)$ is denoted χ_1 and is known. Then the eigenvalue $\lambda_2^2(c)$ is the minimum of the functional (6) over all admissible functions χ orthogonal to χ_1, i.e. with inner product

$$\langle \chi, \chi_1 \rangle \equiv \int_{y_1}^{y_2} \rho_0 \phi(y) y^\nu \chi \chi_1 \, dy = 0$$

(Courant & Hilbert, 1953, p. 401). With the previous definitions of ϵ and g_ϵ retained, construct the function $g_\epsilon^{(2)}$ depending upon a parameter b,

$$g_\epsilon^{(2)} = (y-b)\, g_\epsilon[(y-y_0)^2+\epsilon^2]^{-\frac{1}{2}}$$

and choose b so that $\langle g_\epsilon^{(2)}, \chi_1 \rangle = 0$. Thus

$$b = \frac{\langle y[(y-y_0)^2+\epsilon^2]^{-\frac{1}{2}} g_\epsilon(y), \chi_1 \rangle}{\langle [(y-y_0)^2+\epsilon^2]^{-\frac{1}{2}} g_\epsilon(y), \chi_1 \rangle}$$

$$= y_0 + \frac{\epsilon}{2\pi} \ln \left[\frac{(y_2-y_0)^2+\epsilon^2}{(y_1-y_0)^2+\epsilon^2} \right] + o(\epsilon) \tag{9}$$

† Banks *et al.* (1976) obtain the result for case A by heuristic, but convincing, methods.

as $\epsilon \to 0$. When $g_\epsilon^{(2)}$ is substituted into the functional (6), one again finds the numerator to be $O(\epsilon^0)$ and the denominator is

$$\pi\epsilon^{-1}\left\{1-\frac{1}{4\pi^2}\ln\left(\frac{y_2-y_0}{y_0-y_1}\right)\right\} y_0''\phi(y_0)\,(y_0-y_1)^2\,(y_2-y_0)^2+O(\epsilon^0) \tag{10}$$

so that $\lambda_2^2(c)\to 0$ also as $\epsilon\to 0$ (i.e. as $c\to W_{\min}$). Functions $g_\epsilon^{(n)}$ may be constructed by including $n-1$ adjustable constants to enable one to render $g_\epsilon^{(n)}$ orthogonal to the first $n-1$ eigenfunctions (by a suitable generalization of the idea above): in this way one can show that $\lambda_2^n(c)\to 0$ as $\epsilon\to 0$.

(*b*) *When an extreme value* $W = W_{\text{extreme}}$ *occurs at a boundary point* $\lambda^2(c)\not\to 0$ *as* $c\to W_{\text{extreme}}$. It is not true that $\lambda_1^2(c)\to 0$ as $c\uparrow W_{\min}$ or $c\downarrow W_{\max}$ if the extreme value of W in question occurs at a boundary point: this has been shown in special cases by a combination of numerical methods and matched asymptotic expansions by Banks *et al.* (1976). A neater (and rigorous) demonstration employing the variational principle (6) (and not invoking an asymptotic matching principle) is the following, which builds upon a specific example kindly supplied by Professor L. E. Payne.

We consider only case B: case A is easier to deal with, and a parallel discussion is readily constructed. Consider first the special case $W(y) = By+d$, and assume that the maximum value of $\rho_0\phi$ in $[y_1, y_2]$ is M. It will be shown that $\lambda^2(c)\not\to 0$ as $c\downarrow W_{\max}$ (here $W_{\max} = By_2+d$). The denominator of (6) is

$$\begin{aligned}\int_{y_1}^{y_2}\rho_0\phi\chi^2 y\,dy &\leqslant M\int_{y_1}^{y_2}\chi^2 y\,dy \leqslant \frac{My_2}{c-d}\int_{y_1}^{y_2} y^2\chi^2\frac{d}{dy}\left[B-\frac{c-d}{y}\right]dy\\ &= -\frac{2My_2}{c-d}\int_{y_1}^{y_2}(By+d-c)\,y\chi D_*\chi\,dy\\ &\leqslant \frac{2y_2M^{\frac{1}{2}}}{c-d}\left\{\left(\int_{y_1}^{y_2}My\chi^2\,dy\right)\left(\int_{y_1}^{y_2}(By+d-c)^2(D_*\chi)^2y\,dy\right\}^{\frac{1}{2}},\end{aligned}$$

where integration by parts and the Schwartz inequality have been used. Thus

$$\begin{aligned}M\int_{y_1}^{y_2}y\chi^2\,dy &\leqslant \frac{4My_2^2}{(c-d)^2}\int_{y_1}^{y_2}(By+d-c)^2(D_*\chi)^2y\,dy\\ &\leqslant \frac{4My_2^2}{(c-d)^2}\int_{y_1}^{y_2}(By+d-c)^2Qy\,dy\end{aligned} \tag{11}$$

and therefore

$$\frac{\int_{y_1}^{y_2}(W-c)^2\,Qy\,dy}{\int_{y_1}^{y_2}\rho_0\,\phi(y)\,y\chi^2\,dy}\geqslant\frac{(c-d)^2}{4My_2^2}.$$

If the minimum value of $\rho_0(y) = \alpha \neq 0$, then, for this special linear profile,

$$\frac{\int_{y_1}^{y_2}\rho_0(W-c)^2Qy\,dy}{\int_{y_1}^{y_2}\rho_0\phi y\chi^2\,dy}\geqslant\frac{\alpha(c-d)^2}{4My_2^2}. \tag{12}$$

Hence

$$\lambda^2(c)\geqslant\frac{\alpha(c-d)^2}{4My_2^2} \tag{13}$$

and, since we consider the limit $c \downarrow W_{max} = By_2 + d$, $(c-d)/y_2 = B$,

$$\lim_{c \downarrow W_{max}} \lambda^2(c) \geqslant \frac{\alpha B^2}{4M} > 0. \tag{14}$$

Although this construction is for a specific $W(y)$, it can easily be generalized to any $W(y)$ achieving an overall maximum at $y = y_2$ (and at no other position, excluding flows with W_{max} obtaining at more than one point) with $W'(y_2) \neq 0$. One need only observe that it is possible to choose constants B and d such that $By_2 + d = W(y_2)$ and $c - W(y) \geqslant c - d - By$ for all $y_1 \leqslant y \leqslant y_2$, so that (12)–(14) still follows since

$$\int \rho_0 (W-c)^2 Qy\, dy \geqslant \int \rho_0 (By + d - c)^2 Qy\, dy.$$

The generalized proof above is easily seen to apply also to the cases where $c \uparrow W_{min}$, where the overall minimum W_{min} occurs at $y = y_2$; and by the corresponding limits $c \uparrow W_{min}$, $c \downarrow W_{max}$ where the minimum or maximum of W occurs at $y = y_1$, provided that $W' \neq 0$ at these boundary points.

The results of this section would seem to call into question the conclusion drawn by Chandrasekhar (1961) from his variational principle for c: namely, for any given value of Φ for which a solution to eigenvalue problem (1) (for eigenvalue c) exists with c outside the range of W, then exactly *two* real eigenvalues exist. For, with given functional form $\phi(y)$, Chandrasekhar's conclusion implies that the limit for $\lambda^2(c)$ in (6) as $c \downarrow W_{max}$ is the same as the limit $c \uparrow W_{min}$. But the present results show that the infimum of λ^2 on the lower branch need not be the same as on the upper branch. For example, if one extreme point $W = W_1$ is an interior point, as in subsection (*a*), and the other extreme point $W = W_2$ occurs at a boundary and satisfies the conditions of subsection (*b*), $\lambda^2 \to 0$ as $c \to W_1$, while $\lambda^2 \not\to 0$ as $c \to W_2$. Thus, it seems possible that there may be a range of λ^2 which is associated with only one value of c outside the range of W, as indicated in figure 2.

4. Group velocity

In this section, we establish bounds on the group velocity of propagating neutral modes.

Consider the primary eigenvalue problem for fixed Φ (i.e. λ^2 fixed) and fixed k. According to the last section, there may be zero, one or two real values of c outside the range of W. If real eigenvalues corresponding to a given mode exist, they will vary continuously with the parameter k (holding λ^2 fixed). The function $c_n(k)$ generated in this way for the nth propagation mode, will be designated as the upper branch (for the nth mode) if $c > W_{max}$, and will be designated the lower branch if $c < W_{min}$. When it is necessary to do so, we distinguish the branches by superscripts (u) and (l) as before.

For any mode (the subscript n will be omitted in the remainder of this section with the understanding that reference is made to a single mode), the group velocity is defined to be

$$c_g(k) = \frac{d}{dk}(kc) = c(k) + k\frac{dc}{dk}, \tag{15}$$

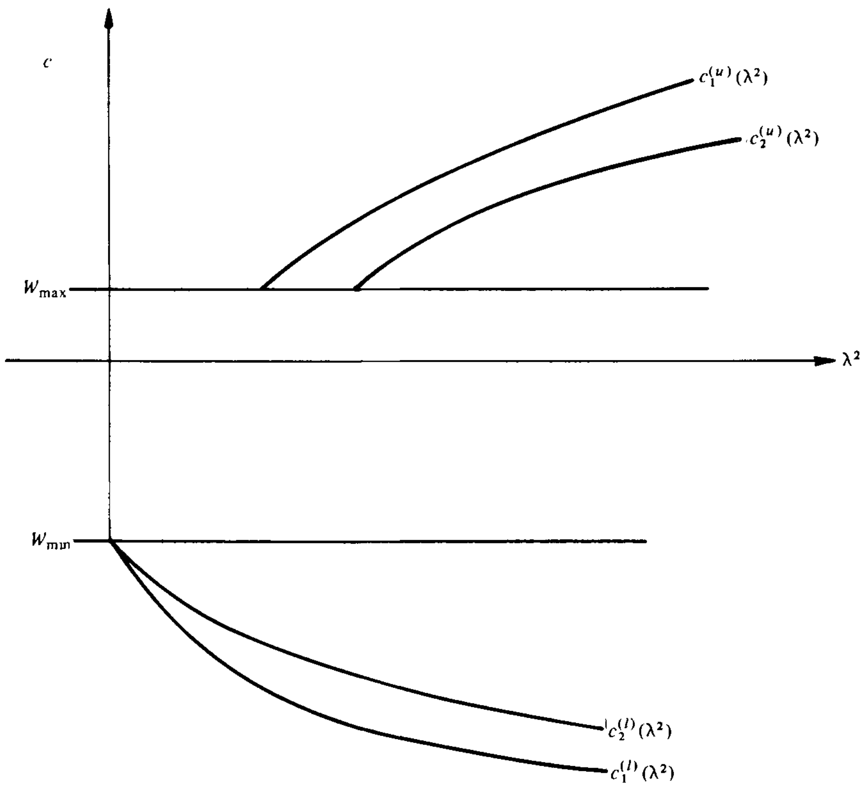

FIGURE 2. Illustrating the results of § 3. Two modes are shown for the lower branch for the case in which the minimum flow speed W_{min} occurs in the interior of the fluid; in this case the lower branch for each mode terminates at $\lambda^2 = 0+$. Two modes of the upper branch are also shown for the case in which the overall maximum flow speed occurs at a boundary; in such cases the branch must terminate at a nonzero value of λ^2. Note that for the case shown here there is a range of λ^2 with only one branch of the function $c(\lambda^2)$.

and we wish to determine how the group velocity corresponding to a given branch of the dispersion relation is related to the phase speed on that branch. To this end we again use the method that produced (5); differentiate (1) with respect to k and let

$$F(y;k) = \frac{\partial \chi(y;k)}{\partial k}.$$

The function F satisfies homogeneous boundary conditions, and an inhomogeneous differential equation with the same operator as (1). Since $\chi(y;k)$ satisfies the homogeneous problem, a solution for $F(y;k)$ is possible only if the following orthogonality condition is satisfied:

$$\int_{y_1}^{y_2} \left\{\frac{dc}{dk}(W-c)\,Q - k(W-c)^2\chi^2\right\} y^\nu \rho_0(y)\,dy = 0, \tag{16}$$

where Q is defined in (5). If (16) is multiplied by k and rearranged, the result

$$k\frac{dc}{dk}\int_{y_1}^{y_2} \rho_0 y^\nu (W-c)\,Q\,dy = k^2\int_{y_1}^{y_2} \rho_0 (W-c)^2\chi^2 y^\nu\,dy \tag{17}$$

may be found.

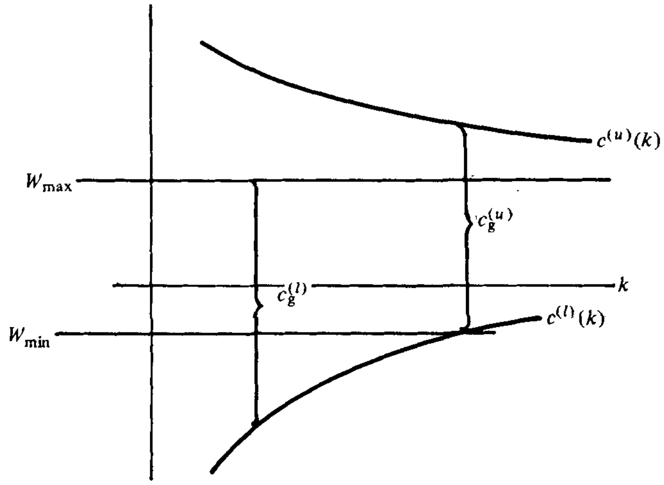

FIGURE 3. Sketch showing the range of possible values of the group velocity $c_g(k)$ for the two branches of the dispersion relation.

By hypothesis, for all admissible values of c, $W-c$ is either strictly positive or strictly negative, depending upon the branch selected for $c(k)$. We therefore may infer from (17) that:

(i) the sign of $k\,dc/dk$ is the same as the sign of $W-c$, thus

$$c_g^{(l)} \geqslant c^{(l)}$$

and
$$c_g^{(u)} \leqslant c^{(u)}.$$

(ii) $k\,dc/dk \to 0$ as $k \to 0$, hence $c_g \to c+0$ if $W-c > 0$ and $c_g \to c-0$ if $W-c < 0$ in the limit $k \to 0$. That $c_g \to c$ in the long wave limit is well known.

We observe that, for $W(y) > 0$, the group velocity for standing waves ($c = 0$) corresponds to $W-c > 0$, hence $c_g \geqslant c$. This result is proved by Benjamin (1962), and is the basis for his criticism of Squire's (1960) vortex breakdown criterion.

Since
$$k^2\chi^2 < Q$$
we may infer from (17) that

$$k\,(dc/dk)\int_{y_1}^{y_2} \rho_0 y^\nu (W-c)\, Q\, dy < \int_{y_1}^{y_2} y^\nu \rho_0 (W-c)^2 Q\, dy,$$

or
$$\int_{y_1}^{y_2} \rho_0 y^\nu (W-c)\,(W-c_g)\, Q\, dy > 0. \tag{18}$$

On the lower branch $W-c^{(l)} > 0$ throughout, and

$$c_g^{(l)}\int_{y_1}^{y_2} \rho_0 y^\nu Q\, dy < \int_{y_1}^{y_2} \rho_0 W y^\nu Q\, dy$$

from which one may conclude that
$$c_g^{(l)} < W_{\max}.$$

By a similar argument, one may conclude that
$$c_g^{(u)} > W_{\min}$$
on the upper branch. We may now summarize (see figure 3 also):

(*a*) for the lower branch, $\quad c^{(l)} \leqslant c_g^{(l)} < W_{\max};$

(*b*) for the upper branch,

$$W_{\min} < c_g^{(u)} \leqslant c^{(u)}.$$

In supercritical flow, both branches have $c > 0$, so that upstream propagation of constant phase surfaces is impossible. The present results show that upstream propagation of wave packets is also prohibited in this case so that in supercritical flow disturbance energy may propagate only in the downstream direction.

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